

# SELF-REGULATION IN CONTINUUM POPULATION MODELS

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**ABSTRACT.** We study the Markov dynamics of an infinite birth-and-death system of point entities placed in  $\mathbb{R}^d$ , in which the constituents disperse and die, also due to competition. Assuming that the dispersal and competition kernels are just continuous and integrable we prove that the evolution of states of this model preserves their sub-Poissonicity, and hence the local self-regulation (suppression of clustering) takes place. Upper bounds for the correlation functions of all orders are also obtained for both long and short dispersals, and for all values of the intrinsic mortality rate  $m \geq 0$ .

## 1. INTRODUCTION

**1.1. The setup.** The aim of the present work is to contribute to the development of the mathematical theory of large systems of living entities, which is a challenging task of modern applied mathematics [2, 3, 19]. Within this task there is the description of the dynamics of individual-based models in which communities of entities appear as configurations of points in some continuous habitat, see [4, 18, 20, 21]. In particular, these can be birth-and-death models, in which the dynamics amounts to the appearance (birth) and disappearance (death) of the constituents. The fact that the disappearance of a given entity is related to its interaction with the existing community is interpreted as *competition* between the entities.

In the simplest birth-and-death models, the system is finite and the state space is  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . That is, in state  $n \in \mathbb{N}_0$  the system consists of  $n$  entities, which is the complete characterization of the state. Then the only observed result of the trade-off between the appearance and disappearance is the dynamics of the number of entities in the whole population. The theory of such models goes back to works by A. Kolmogorov and W. Feller, see [7, Chapter XVII] and, e.g., [1, 22], for a more recent account of the related concepts and results. Therein the time evolution of the probability of having  $n$  entities is obtained by solving the Kolmogorov equation with a tridiagonal infinite matrix on the right-hand side. Its entries are expressed in terms of the birth and death rates  $\lambda_n$  and  $\mu_n$ , respectively. If the increase of  $\lambda_n$  and  $\mu_n$  is controlled by affine functions of  $n$ , the solution of the Kolmogorov equation is given by a stochastic semigroup, see, e.g., [1] and the literature quoted in this work. However, if  $\lambda_n$  and  $\mu_n \lambda_n$  increase faster than  $n$ , this is no more the case. For infinite systems, the very definition of the Kolmogorov equation gets problematic as the mentioned birth and death rates get infinite. Usually such systems are considered in some spatial habitat and the parameters that describe the interactions between the entities are space-dependent. Then then along with the global characteristics the system acquires a local time-dependent structure. The mentioned trade-off may affect this structure with or without affecting the global dynamics of the population. This relates also to the models with traits other than the spatial position.

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In this work we continue dealing with the model introduced in [4, 5, 18]. Here the spatial habitat is the Euclidean space  $\mathbb{R}^d$ ,  $d \geq 1$ , equipped with the usual outfit of mathematical structures. Then the phase space is the set  $\Gamma$  of all subsets  $\gamma \subset \mathbb{R}^d$  such that the set  $\gamma_\Lambda := \gamma \cap \Lambda$  is finite whenever  $\Lambda \subset \mathbb{R}^d$  is compact. For each such  $\Lambda$ , one defines the counting map  $\Gamma \ni \gamma \mapsto |\gamma_\Lambda| := \#\{\gamma \cap \Lambda\}$ , where the latter denotes cardinality. Thereby, one introduces the subsets  $\Gamma^{\Lambda,n} := \{\gamma \in \Gamma : |\gamma_\Lambda| = n\}$ ,  $n \in \mathbb{N}_0$ , and equips  $\Gamma$  with the  $\sigma$ -field generated by all such  $\Gamma^{\Lambda,n}$ . This allows for considering probability measures on  $\Gamma$  as states of the system. Among them there are Poissonian states in which the entities are independently distributed over  $\mathbb{R}^d$ , see [13, Chapter 2]. They may serve as reference states for studying correlations between the positions of the entities. For the homogeneous Poisson measure  $\pi_\varkappa$  with density  $\varkappa > 0$  and every compact  $\Lambda$ , one has

$$\pi_\varkappa(\Gamma^{\Lambda,n}) = (\varkappa V(\Lambda))^n \exp(-\varkappa V(\Lambda)) / n!, \quad n \in \mathbb{N}_0, \quad (1.1)$$

where  $V(\Lambda)$  denotes Lebesgue's measure (volume) of  $\Lambda$ . Let us agree to call a state,  $\mu$ , *sub-Poissonian* (cf. Definition 2.1 and Remark 2.2 below) if, for each compact  $\Lambda \subset \mathbb{R}^d$ , the following holds

$$\forall n \in \mathbb{N}_0 \quad \mu(\Gamma^{\Lambda,n}) \leq C_\Lambda \varkappa_\Lambda^n / n!, \quad (1.2)$$

with some positive constants  $C_\Lambda$  and  $\varkappa_\Lambda$ . By the virtue of this definition, the sub-Poissonian states are characterized by the lack of *heavy tails* or *clustering*. That is, the entities in such a state are either independent in taking their positions or 'prefer' to stay away of each other.

The counting map  $\Gamma \ni \gamma \mapsto |\gamma|$  can also be defined for  $\Lambda = \mathbb{R}^d$ . Then the set of *finite* configurations

$$\Gamma_0 := \bigcup_{n \in \mathbb{N}_0} \{\gamma \in \Gamma : |\gamma| = n\} \quad (1.3)$$

is clearly measurable. In a state with the property  $\mu(\Gamma_0) = 1$ , the system is ( $\mu$ -almost surely) *finite*. By (1.1)  $\pi_\varkappa(\Gamma_0) = 0$ , hence the system in state  $\pi_\varkappa$  is infinite in the same sense. A nonhomogeneous Poisson measure  $\pi_\varrho$ , characterized by density  $\varrho : \mathbb{R}^d \rightarrow [0, +\infty)$ , satisfies (1.1) with  $\varkappa V(\Lambda)$  replaced by  $\int_\Lambda \varrho(x) dx$ . Then either  $\pi_\varrho(\Gamma_0) = 1$  or  $\pi_\varrho(\Gamma_0) = 0$ , depending on whether or not  $\varrho$  is globally integrable. The use of infinite configurations for modeling large finite populations is as a rule justified, see, e.g., [6], by the argument that in such a way one gets rid of the boundary and size effects. Note that a finite system with dispersal – like the one specified in (1.6) and (1.7) below – being placed in a noncompact habitat always disperse to fill its empty parts, and thus is *developing*. Infinite configurations are supposed to model *developed* populations. In this work, we shall consider infinite systems and hence deal with states  $\mu$  such that  $\mu(\Gamma_0) = 0$ .

**1.2. The article overview.** To characterize states on  $\Gamma$  one employs *observables* – appropriate functions  $F : \Gamma \rightarrow \mathbb{R}$ . Their evolution is obtained from the Kolmogorov equation

$$\frac{d}{dt} F_t = L F_t, \quad F_t|_{t=0} = F_0, \quad t > 0, \quad (1.4)$$

where the generator  $L$  specifies the model. The states' evolution is then obtained from the Fokker–Planck equation

$$\frac{d}{dt} \mu_t = L^* \mu_t, \quad \mu_t|_{t=0} = \mu_0, \quad (1.5)$$

related to that in (1.4) by the duality  $\mu_t(F_0) = \mu_0(F_t)$ , where

$$\mu(F) := \int_{\Gamma} F(\gamma) \mu(d\gamma).$$

The model discussed in this work is specified by the following

$$\begin{aligned} (LF)(\gamma) &= \sum_{x \in \gamma} E^-(x, \gamma \setminus x) [F(\gamma \setminus x) - F(\gamma)] \\ &+ \int_{\mathbb{R}^d} E^+(x, \gamma) [F(\gamma \cup x) - F(\gamma)] dx, \end{aligned} \quad (1.6)$$

where  $E^+(x, \gamma)$  and  $E^-(x, \gamma)$  are state-dependent birth and death rates, respectively. They have the following forms

$$E^+(x, \gamma) = \sum_{y \in \gamma} a^+(x - y), \quad (1.7)$$

$$E^-(x, \gamma) = m + \sum_{y \in \gamma} a^-(x - y), \quad (1.8)$$

where  $a^+ \geq 0$  and  $a^- \geq 0$  are the *dispersal* and *competition kernels*, respectively,  $m \geq 0$  is the intrinsic mortality rate. This model was introduced in [4, 5, 18]. Its recent study can be found in [11, 14, 17], see also older works [9, 10]. For the kernels  $a^\pm$ , one has the following possibilities:

- (i) (*short dispersal*) there exists  $\theta > 0$  such that  $a^-(x) \geq \theta a^+(x)$  for all  $x \in \mathbb{R}^d$ ;
- (ii) (*long dispersal*) for each  $\theta > 0$ , there exists  $x \in \mathbb{R}^d$  such that  $a^-(x) < \theta a^+(x)$ .

In case (i),  $a^+$  decays faster than  $a^-$ , and hence each daughter entity can ‘kill’ her mother as well as can be ‘killed’ by her. Such models are usually employed to described the dynamics of cell communities, see [8], where the dispersal is just the cell division. An instance of the short dispersal is given by  $a^+$  with finite range, i.e.,  $a^+(x) \equiv 0$  for all  $|x| \geq r$ , and  $a^-(x) > 0$  for such  $x$ . In case (ii),  $a^-$  decays faster than  $a^+$ , and hence some of the offsprings can be out of reach of their parents. Models of this kind can be adequate, e.g., in plant ecology with the long-range dispersal of seeds, cf. [20].

In this article, the model parameters are supposed to satisfy the following.

**Assumption 1.1.** *The kernels  $a^\pm$  in (1.7) and (1.8) are continuous and belong to  $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ .*

According to this we set

$$\langle a^\pm \rangle = \int_{\mathbb{R}^d} a^\pm(x) dx, \quad \|a^\pm\| = \sup_{x \in \mathbb{R}^d} a^\pm(x). \quad (1.9)$$

For the model with  $E^+$  as in (1.7), in [14] we have constructed the evolution of states  $\mu_0 \mapsto \mu_t$ ,  $t > 0$ , which preserves the sub-Poissonicity under a certain condition on the dispersal and competition kernels. In this work, by means of the result proved in Lemma 3.1 we eliminate this restriction and prove that the local self-regulation in this model occurs (Theorem 2.5) if  $a^\pm$  satisfy just a minimum set of assumptions, see Assumption 1.1 and also the corresponding comments in subsection 2.3. For the migration model specified in (??) and (1.8), we prove that the evolution of states  $\mu_0 \mapsto \mu_t$ ,  $t > 0$ , exists (Theorem ??) and is such that  $\mu_t(N_\Lambda^n) \leq C_\Lambda^{(n)}$  for each  $t > 0$ . We do this as follows. First, assuming the existence of the evolution  $\mu_0 \mapsto \mu_t$ , we prove that it is characterized by the global self-regulation as just mentioned, see Theorem 2.6 and Lemma ??. Then we prove the existence of this evolution.

The structure of the article is as follows. In Section 2, we introduce the necessary technicalities and then formulate the results: Theorems 2.5 and 2.6. Thereafter, we make a number of comments to these results and compare them with the facts known for similar objects. In Section 3, we present the proof of the both mentioned statements assuming the existence of the evolution of states in the migration model. In Section 4, we prove the latter fact.

## 2. PRELIMINARIES AND THE RESULTS

By  $\mathcal{B}(\mathbb{R})$  we denote the sets of all Borel subsets of  $\mathbb{R}$ . The configuration space  $\Gamma$  is equipped with the vague topology, see [?, 15], and thus with the corresponding Borel  $\sigma$ -field  $\mathcal{B}(\Gamma)$ , which makes it a standard Borel space. Note that  $\mathcal{B}(\Gamma)$  is exactly the  $\sigma$ -field generated by the sets  $\Gamma^{\Lambda,n}$ , mentioned in Introduction. By  $\mathcal{P}(\Gamma)$  we denote the set of all probability measures on  $(\Gamma, \mathcal{B}(\Gamma))$ .

**2.1. Correlation functions.** Like in [9, 11, 14], the evolution of states will be described by means of correlation functions. To explain the essence of this approach let us consider the set of all compactly supported continuous functions  $\theta : \mathbb{R}^d \rightarrow (-1, 0]$ . For a state,  $\mu$ , its *Bogoliubov* functional, cf. [16], is

$$B_\mu(\theta) = \int_\Gamma \prod_{x \in \gamma} (1 + \theta(x)) \mu(d\gamma), \quad (2.1)$$

with  $\theta$  running through the mentioned set of functions. For the homogeneous Poisson measure  $\pi_\varkappa$ , it takes the form

$$B_{\pi_\varkappa}(\theta) = \exp \left( \varkappa \int_{\mathbb{R}^d} \theta(x) dx \right).$$

**Definition 2.1.** *The set of states  $\mathcal{P}_{\text{exp}}(\Gamma)$  is defined as that containing all those states  $\mu \in \mathcal{P}(\Gamma)$  for which  $B_\mu$  can be continued, as a function of  $\theta$ , to an exponential type entire function on  $L^1(\mathbb{R}^d)$ .*

It can be shown that a given  $\mu$  belongs to  $\mathcal{P}_{\text{exp}}(\Gamma)$  if and only if its functional  $B_\mu$  can be written down in the form

$$B_\mu(\theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} k_\mu^{(n)}(x_1, \dots, x_n) \theta(x_1) \cdots \theta(x_n) dx_1 \cdots dx_n, \quad (2.2)$$

where  $k_\mu^{(n)}$  is the  $n$ -th order correlation function of  $\mu$ . It is a symmetric element of  $L^\infty((\mathbb{R}^d)^n)$  for which

$$\|k_\mu^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} \leq C \exp(\vartheta n), \quad n \in \mathbb{N}_0, \quad (2.3)$$

with some  $C > 0$  and  $\vartheta \in \mathbb{R}$ . Note that  $k_{\pi_\varkappa}^{(n)}(x_1, \dots, x_n) = \varkappa^n$ . Note also that (2.2) resembles the Taylor expansion of the characteristic function of a probability measure. In view of this,  $k_\mu^{(n)}$  are also called (factorial) *moment functions*, cf. [4, 5, 18].

**Remark 2.2.** *By (2.3) each  $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$  satisfies (1.2) and hence is a sub-Poissonian state.*

Recall that  $\Gamma_0$  – the set of all finite  $\gamma \in \Gamma$  defined in (1.3) – is an element of  $\mathcal{B}(\Gamma)$ . A function  $G : \Gamma_0 \rightarrow \mathbb{R}$  is  $\mathcal{B}(\Gamma)/\mathcal{B}(\mathbb{R})$ -measurable, see [11], if and only if, for each  $n \in \mathbb{N}$ , there exists a symmetric Borel function  $G^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$  such that

$$G(\eta) = G^{(n)}(x_1, \dots, x_n), \quad \text{for } \eta = \{x_1, \dots, x_n\}. \quad (2.4)$$

**Definition 2.3.** A measurable function  $G : \Gamma_0 \rightarrow \mathbb{R}$  is said to have bounded support if: (a) there exists  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$  such that  $G(\eta) = 0$  whenever  $\eta \cap (\mathbb{R}^d \setminus \Lambda) \neq \emptyset$ ; (b) there exists  $N \in \mathbb{N}_0$  such that  $G(\eta) = 0$  whenever  $|\eta| > N$ . By  $\Lambda(G)$  and  $N(G)$  we denote the smallest  $\Lambda$  and  $N$  with the properties just mentioned. By  $B_{bs}(\Gamma_0)$  we denote the set of all such functions.

The Lebesgue-Poisson measure  $\lambda$  on  $(\Gamma_0, \mathcal{B}(\Gamma_0))$  is defined by the following formula

$$\int_{\Gamma_0} G(\eta) \lambda(d\eta) = G(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n, \quad (2.5)$$

holding for all  $G \in B_{bs}(\Gamma_0)$ . Like in (2.4), we introduce  $k_\mu : \Gamma_0 \rightarrow \mathbb{R}$  such that  $k_\mu(\eta) = k_\mu^{(n)}(x_1, \dots, x_n)$  for  $\eta = \{x_1, \dots, x_n\}$ ,  $n \in \mathbb{N}$ . We also set  $k_\mu(\emptyset) = 1$ . With the help of the measure introduced in (2.5), the expressions for  $B_\mu$  in (2.1) and (2.2) can be combined into the following formulas

$$\begin{aligned} B_\mu(\theta) &= \int_{\Gamma_0} k_\mu(\eta) \prod_{x \in \eta} \theta(x) \lambda(d\eta) =: \int_{\Gamma_0} k_\mu(\eta) e(\eta; \theta) \lambda(d\eta) \\ &= \int_{\Gamma} \prod_{x \in \gamma} (1 + \theta(x)) \mu(d\gamma) =: \int_{\Gamma} F_\theta(\gamma) \mu(d\gamma). \end{aligned} \quad (2.6)$$

Thereby, one can transform the action of  $L$  on  $F$ , see (1.6), to the action of  $L^\Delta$  on  $k_\mu$  according to the rule

$$\int_{\Gamma} (LF_\theta)(\gamma) \mu(d\gamma) = \int_{\Gamma_0} (L^\Delta k_\mu)(\eta) e(\eta; \theta) \lambda(d\eta). \quad (2.7)$$

This will allow us to pass from (1.5) to the corresponding Cauchy problem for the correlation functions

$$\frac{d}{dt} k_t = L^\Delta k_t, \quad k_t|_{t=0} = k_{\mu_0}. \quad (2.8)$$

For the Bolker-Pacala model specified in (1.6) and (1.7), (1.8), by (2.7) the action of  $L^\Delta$  looks as follows, cf. [11, 14],

$$\begin{aligned} (L^\Delta k)(\eta) &= (L^{\Delta, -} k)(\eta) + \sum_{x \in \eta} E^+(x, \eta \setminus x) k(\eta \setminus x) \\ &+ \int_{\mathbb{R}^d} \sum_{x \in \eta} a^+(x - y) k(\eta \setminus x \cup y) dy, \end{aligned} \quad (2.9)$$

where

$$(L^{\Delta, -} k)(\eta) := -E^-(\eta) k(\eta) - \int_{\mathbb{R}^d} \left( \sum_{y \in \eta} a^-(x - y) \right) k(\eta \cup x) dx, \quad (2.10)$$

and

$$E^-(\eta) := \sum_{x \in \eta} E^-(x, \eta \setminus x). \quad (2.11)$$

For the migration model specified in (1.6) and (??), (1.8), by (2.7) we obtain

$$(L^\Delta k)(\eta) = (L^{\Delta, -} k)(\eta) + \sum_{x \in \eta} b(x) k(\eta \setminus x), \quad (2.12)$$

with the same  $L^{\Delta, -}$  as in (2.10). In the next subsection, we introduce the spaces where we are going to define the problems (2.8).

**2.2. The statements.** By (2.2) and (2.6), it follows that  $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$  implies

$$|k_\mu(\eta)| \leq C \exp(\vartheta|\eta|),$$

holding for  $\lambda$ -almost all  $\eta \in \Gamma_0$ , some  $C > 0$ , and  $\vartheta \in \mathbb{R}$ . In view of this, we set

$$\mathcal{K}_\vartheta := \{k : \Gamma_0 \rightarrow \mathbb{R} : \|k\|_\vartheta < \infty\}, \quad (2.13)$$

where

$$\|k\|_\vartheta = \text{ess sup}_{\eta \in \Gamma_0} \{ |k_\mu(\eta)| \exp(-\vartheta|\eta|) \}. \quad (2.14)$$

Clearly, (2.13) and (2.14) define a Banach space. In the following, we use the ascending scale of such spaces  $\mathcal{K}_\vartheta$ ,  $\vartheta \in \mathbb{R}$ , with the property

$$\mathcal{K}_\vartheta \hookrightarrow \mathcal{K}_{\vartheta'}, \quad \vartheta < \vartheta', \quad (2.15)$$

where  $\hookrightarrow$  denotes continuous embedding.

For  $G \in B_{\text{bs}}(\Gamma)$ , we set

$$(KG)(\gamma) = \sum_{\eta \in \gamma} G(\eta), \quad (2.16)$$

where  $\sum$  denotes the summation is taken over all finite subsets. It satisfies, see Definition 2.3,

$$|(KG)(\gamma)| \leq (1 + |\gamma \cap \Lambda(G)|)^{N(G)}.$$

The latter means that  $\mu(KG) < \infty$  for each  $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ . By (2.6) this yields

$$\langle\langle G, k_\mu \rangle\rangle := \int_{\Gamma_0} G(\eta) k_\mu(\eta) \lambda(d\eta) = \mu(KG) < \infty. \quad (2.17)$$

Set

$$B_{\text{bs}}^*(\Gamma_0) = \{G \in B_{\text{bs}}(\Gamma_0) : (KG)(\gamma) \geq 0 \text{ for all } \gamma \in \Gamma\}. \quad (2.18)$$

By [15, Theorems 6.1 and 6.2 and Remark 6.3] one can prove the next statement.

**Proposition 2.4.** *Let a measurable function  $k : \Gamma_0 \rightarrow \mathbb{R}$  have the following properties:*

$$\begin{aligned} (a) \quad & \langle\langle G, k \rangle\rangle \geq 0, \quad \text{for all } G \in B_{\text{bs}}^*(\Gamma_0); \\ (b) \quad & k(\emptyset) = 1; \quad (c) \quad k(\eta) \leq C^{|\eta|}, \end{aligned} \quad (2.19)$$

*with (c) holding for some  $C > 0$  and  $\lambda$ -almost all  $\eta \in \Gamma_0$ . Then there exists a unique state  $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$  for which  $k$  is the correlation function.*

Set, cf. (2.18),

$$\mathcal{K}_\vartheta^* = \{k \in \mathcal{K}_\vartheta : \langle\langle G, k \rangle\rangle \geq 0 \text{ for all } G \in B_{\text{bs}}^*(\Gamma_0)\}, \quad (2.20)$$

which is a subset of the cone

$$\mathcal{K}_\vartheta^+ = \{k \in \mathcal{K}_\vartheta : k(\eta) \geq 0 \text{ for } \lambda - \text{almost all } \eta \in \Gamma_0\}. \quad (2.21)$$

By Proposition 2.4 it follows that each  $k \in \mathcal{K}_\vartheta^*$  such that  $k(\emptyset) = 1$  is the correlation function of a unique state  $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ . Then we define

$$\mathcal{K} = \bigcup_{\vartheta \in \mathbb{R}} \mathcal{K}_\vartheta, \quad \mathcal{K}^* = \bigcup_{\vartheta \in \mathbb{R}} \mathcal{K}_\vartheta^*.$$

As a sum of Banach spaces, the linear space  $\mathcal{K}$  is equipped with the corresponding inductive topology that turns it into a locally convex space.

For each  $\vartheta \in \mathbb{R}$  and  $\vartheta' > \vartheta$ , the expressions in (2.9), (2.10) and (2.12) can be used to define the corresponding bounded linear operators  $L_{\vartheta', \vartheta}^\Delta$  acting from  $\mathcal{K}_\vartheta$  to  $\mathcal{K}_{\vartheta'}$ . Their

operator norms can be estimated similarly as in [14, eqs. (3.11), (3.13)], which yields, cf. (1.9),

(a) Bolker – Pacala model:

$$\|L_{\vartheta'\vartheta}^\Delta\| \leq \frac{4(\|a^+\| + \|a^-\|)}{e^2(\vartheta' - \vartheta)^2} + \frac{\langle a^+ \rangle + \langle a^- \rangle e^{\vartheta'}}{e(\vartheta' - \vartheta)}, \quad (2.22)$$

(b) migration model:

$$\|L_{\vartheta'\vartheta}^\Delta\| \leq \frac{4\|a^-\|}{e^2(\vartheta' - \vartheta)^2} + \frac{\|b\|e^{-\vartheta} + \langle a^- \rangle e^{\vartheta'}}{e(\vartheta' - \vartheta)}. \quad (2.23)$$

By means of the collection  $\{L_{\vartheta'\vartheta}^\Delta\}$  we introduce the corresponding continuous linear operators acting on  $\mathcal{K}$ , and thus define the corresponding Cauchy problems (2.8) in this space. By their (global in time) solutions we will mean continuously differentiable functions  $[0, +\infty) \ni t \mapsto k_t \in \mathcal{K}$  such that both equalities in (2.8) hold. Our results are given in the following statements, both based on Assumption 1.1.

**Theorem 2.5** (Bolker-Pacala model). *For each  $\mu_0 \in \mathcal{P}_{\text{exp}}(\Gamma)$ , the problem in (2.8) with  $L^\Delta : \mathcal{K} \rightarrow \mathcal{K}$  as in (2.9) – (2.11) and (2.22) has a unique solution which lies in  $\mathcal{K}^*$  and is such that  $k_t(\emptyset) = 1$  for all  $t > 0$ . Therefore, for each  $t > 0$ , there exists a unique state  $\mu_t \in \mathcal{P}_{\text{exp}}(\Gamma)$  such that  $k_t = k_{\mu_t}$ .*

**Theorem 2.6** (Migration model). *For each  $\mu_0 \in \mathcal{P}_{\text{exp}}(\Gamma)$ , the problem in (2.8) with  $L^\Delta : \mathcal{K} \rightarrow \mathcal{K}$  as in (2.10), (2.12) and (2.23) has a unique solution which lies in  $\mathcal{K}^*$  and is such that  $k_t(\emptyset) = 1$  for all  $t > 0$ . Therefore, for each  $t > 0$ , there exists a unique state  $\mu_t \in \mathcal{P}_{\text{exp}}(\Gamma)$  such that  $k_t = k_{\mu_t}$ . Moreover, these states  $\mu_t$  have the property: for every  $n \in \mathbb{N}$  and  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ , the following holds, cf. (??),*

$$\forall t > 0 \quad \mu_t(N_\Lambda^n) \leq C_\Lambda^{(n)}.$$

**2.3. Comments and comparison.** For  $a^- \equiv 0$ , both models considered in this work get exactly soluble. The Bolker-Pacala version with  $E^-(x, \gamma) = m$  ( $m \geq 0$  being the intrinsic mortality rate) is known as the continuum contact model – see [9], and also the discussion in [14, Introduction] and the literature cited therein. In this model, the evolution  $\mu_0 \mapsto \mu_t$  does not preserve the class  $\mathcal{P}_{\text{exp}}(\Gamma)$ , cf. [9, eq. (3.5), page 303]. For  $m \geq \langle a^+ \rangle$ , the correlation functions remain bounded in time. That is, the global regulation is achieved at the expense of large intrinsic mortality. Moreover, the system dies out if  $m < \langle a^+ \rangle$ .

The migration version of (1.6) with  $E^-(x, \gamma) = m$  is known as the Surgailis model, see [?] and the discussion in [?]. See also its solution in (??) below obtained for  $E^- \equiv 0$ . For this model with  $m > 0$ , the large-time limits of the correlation functions are Poissonian with the density  $\varrho(x) = b(x)/m$ . If the initial state is Poissonian with density  $\varrho_0(x)$ , and if  $m = 0$ , the state  $\mu_t$  is also Poissonian with the density  $\varrho_t(x) = \varrho_0(x) + b(x)t$ , cf. (??) below. That is, in this model the global regulation is possible only if  $m(x) \geq m > 0$  for all  $x \in \mathbb{R}^d$ .

Now we give more specific comments to each of the models.

**2.3.1. The Bolker-Pacala model.** As follows from our Theorem 2.5, adding competition to the continuum contact model mentioned above yields the local self-regulation – no matter how long the dispersal is and how local is the competition. Also their magnitudes do not matter for the very fact of the self-regulation. A sufficient condition under which the property stated in Theorem 2.5 (as well as in [14, Theorem 3.3]) holds,

cf. [14, eq. (3.5)], in the present notations is

$$\omega|\eta| + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^-(x-y) \geq \theta \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^+(x-y), \quad (2.24)$$

holding for some  $\omega \geq 0$  and  $\theta > 0$ , and  $\lambda$ -almost all  $\eta \in \Gamma_0$ . Recall that  $|\eta|$  stands for the cardinality of  $\eta \in \Gamma_0$ . In the short dispersal case (studied in [11]) the condition in (2.24) readily holds with  $\omega = 0$ . Then the most intriguing question here is whether it can hold in the long dispersal case. In [14, Proposition 3.7], it was shown that measurable  $a^+$  and  $a^-$  satisfy (2.24) with some  $\omega$  and  $\theta$  if  $a^-(x)$  is separated away from zero for  $|x| < r$  with some  $r > 0$ , and  $a^+(x) \equiv 0$  for  $|x| \geq R$  with some  $R > 0$  with the possibility  $R > r$ . Another choice of  $a^+$  and  $a^-$  satisfying (2.24) can be, see [14, Proposition 3.8],

$$a^\pm(x) = \frac{c_\pm}{(2\pi\sigma_\pm^2)^{d/2}} \exp\left(-\frac{1}{2\sigma_\pm^2}|x|^2\right),$$

with all possible values of the parameters  $c_\pm > 0$  and  $\sigma_\pm > 0$ . An important example of  $a^\pm$  which both Propositions 3.7 and 3.8 of [14] do not cover is  $a^-$  having finite range and  $a^+$  being Gaussian as above. The novelty of our present (rather unexpected) result is that (2.24) is satisfied *for any*  $a^+$  and  $a^-$  as in Assumption 1.1, and hence the local self-regulation is achieved by applying any kind of competition. Does not matter how weak and short-ranged. Finally, we remark that for this model the conditions of boundedness and continuity of  $a^-$  can be relaxed. As follows from the proof of Lemma 3.1 below, like in [14] it is enough to assume that  $a^-$  is measurable and separated away from zero in some ball. As for  $a^+$ , it is enough to have a continuous  $\tilde{a}^+ \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  such that  $\tilde{a}^+(x) \geq a^+(x)$  for almost all  $x$ .

### 3. THE PROOF THEOREMS 2.5

In [14, Theorem 3.3], it was proved that the evolution in question exists whenever the kernels  $a^\pm$  are bounded and integrable, and satisfy (2.24) with some  $\omega$  and  $\theta$ . Thus, the proof of Theorem 2.5 relies upon proving the following statement.

**Lemma 3.1.** *Let  $a^\pm$  satisfy Assumption 1.1. Then one finds  $\omega \geq 0$  and  $\theta > 0$  such that (2.24) holds for all  $\eta \in \Gamma_0$ .*

*Proof.* Since  $a^+$  is Riemann integrable, for an arbitrary  $\varepsilon > 0$ , one can divide  $\mathbb{R}^d$  into equal cubic cells  $E_l$ ,  $l \in \mathbb{N}$ , of small enough side  $h > 0$  such that the following holds, see (1.9),

$$h^d \sum_{l=1}^{\infty} a_l^+ \leq \langle a^+ \rangle + \varepsilon, \quad a_l^+ := \sup_{x \in E_l} a^+(x). \quad (3.1)$$

Given  $r > 0$  and  $x \in \mathbb{R}^d$ , we set  $K_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}$  and

$$a_r^- = \inf_{x \in K_{2r}(0)} a^-(x). \quad (3.2)$$

Then we fix  $\varepsilon$  and pick  $r > 0$  such that  $a_r^- > 0$ . In the following,  $r$ ,  $h$  and  $\varepsilon$  are fixed.

The proof of the lemma will be done by the induction in the number of points in  $\eta$ . To do this we rewrite (2.24) in the form

$$U_\theta(\eta) := \omega|\eta| + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^-(x-y) - \theta \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^+(x-y) \geq 0. \quad (3.3)$$



For some  $x \in \eta$ , consider

$$\begin{aligned} U_\theta(x, \eta \setminus x) &:= U_\theta(\eta) - U_\theta(\eta \setminus x) \\ &= \omega + 2 \left( \sum_{y \in \eta \setminus x} a^-(x-y) - \theta \sum_{y \in \eta \setminus x} a^+(x-y) \right). \end{aligned} \quad (3.4)$$

Let  $c_d$  be the volume of the unit ball  $K_1$  and  $\Delta(d)$  be the packing constant for the rigid balls in  $\mathbb{R}^d$ , cf. [12]. Set

$$\delta(a^+) = \max \{ \|a^+\|; (\langle a^+ \rangle + \varepsilon) g_d(h, r) \}, \quad (3.5)$$

$$g_d(h, r) = \frac{\Delta(d)}{c_d} \left( \frac{h+2r}{hr} \right)^d,$$

and assume that  $\omega$  and  $\theta$  satisfy the following, cf. (3.2),

$$\theta \leq \min \left\{ \frac{\omega}{2\delta(a^+)}; \frac{a_r^-}{\delta(a^+)} \right\}. \quad (3.6)$$

Let us show that:

- (a) for each  $\eta = \{x, y\}$ , (3.6) yields (3.3);
- (b) for each  $\eta$ , one finds  $x \in \eta$  such that  $U_\theta(x, \eta \setminus x) \geq 0$  whenever (3.6) holds.

If both (a) and (b) hold, then (3.3) will follow from (3.4) by the induction in  $|\eta|$ . To prove (a) we write

$$\begin{aligned} U_\theta(\{x, y\}) &= 2\omega + 2a^-(x-y) - 2\theta a^+(x-y) \\ &\geq (\omega - 2\theta \|a^+\|) + 2a^-(x-y) \geq 0, \end{aligned}$$

with the latter estimate following by (3.6) and (3.5). To prove (b), for  $y \in \eta$ , we set

$$s = \max_{y \in \eta} |\eta \cap K_{2r}(y)|. \quad (3.7)$$

Let also  $x \in \eta$  be such that  $|\eta \cap K_{2r}(x)| = s$ . For this  $x$ , by  $E_l(x)$ ,  $l \in \mathbb{N}$ , we denote the corresponding translates of  $E_l$  which appear in (3.1). Set  $\eta_l = \eta \cap E_l(x)$  and let  $l_* \in \mathbb{N}$  be such that  $\eta \subset \cup_{l \leq l_*} E_l(x)$ , which is possible since  $\eta$  is finite. For a given  $l$ , a subset  $\xi_l \subset \eta_l$  is called  $r$ -admissible if for each distinct  $y, z \in \xi_l$ , one has that  $K_r(y) \cap K_r(z) = \emptyset$ . Such a subset  $\xi_l$  is called maximal  $r$ -admissible if  $|\xi_l| \geq |\xi'_l|$  for any other  $r$ -admissible  $\xi'_l$ . It is clear that

$$\eta_l \subset \bigcup_{z \in \xi_l} K_{2r}(z). \quad (3.8)$$

Otherwise, one finds  $y \in \eta_l$  such that  $|y - z| \geq 2r$ , for each  $z \in \xi_l$ , which yields that  $\xi_l$  is not maximal. Since all the balls  $K_r(z)$ ,  $z \in \xi_l$ , are contained in the  $h$ -extended cell

$$E_l^h(x) := \{y \in \mathbb{R}^d : \inf_{z \in E_l(x)} |y - z| \leq h\},$$

their maximum number – and hence  $|\xi_l|$  – can be estimated as follows

$$|\xi_l| \leq \Delta(d) V(E_l^h(x)) / c_d r^d = h^d \frac{\Delta(d)}{c_d} \left( \frac{h+2r}{hr} \right)^d = h^d g_d(h, r), \quad (3.9)$$

where  $c_d$  and  $\Delta(d)$  are as in (3.5). Then by (3.7) and (3.8) we get

$$\sum_{y \in \eta \setminus x} a^+(x-y) \leq \sum_{l=1}^{l_*} \sum_{z \in \xi_l} \sum_{y \in K_{2r}(z) \cap \eta_l} a_l^+.$$

The cardinality of  $K_{2r}(z) \cap \eta_l$  does not exceed  $s$ , see (3.7), whereas the cardinality of  $\xi_l$  satisfies (3.9). Then

$$\sum_{y \in \eta \setminus x} a^+(x - y) \leq sg_d(h, r) \sum_{l=1}^{\infty} a_l^+ h^d \leq sg_d(h, r)(\langle a^+ \rangle + \varepsilon) \leq s\delta(a^+). \quad (3.10)$$

On the other hand, by (3.2) and (3.7) we get

$$\sum_{y \in \eta \setminus x} a^-(x - y) \geq \sum_{y \in (\eta \setminus x) \cap K_{2r}(x)} a^-(x - y) \geq (s - 1)a_r^-.$$

We use this estimate and (3.10) in (3.4) and obtain

$$U_\theta(x, \eta \setminus x) \geq 2\delta(a^+) \left[ \left( \frac{\omega}{2\delta(a^+)} - \theta \right) + (s - 1) \left( \frac{a_r^-}{\delta(a^+)} - \theta \right) \right] \geq 0,$$

see (3.6). Thus, claim (b) also holds, which completes the proof.  $\square$

Now the proof of Theorem 2.5 follows by [14, Theorem 3.3] and Lemma 3.1 just proved.

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#### REFERENCES

- [1] J. Banasiak, M. Lachowicz, M. Moszyński, Semigroups for generalized birth-and-death equations in  $\ell^p$  spaces, *Semigroup Forum* **73** (2006) 175–193.
- [2] N. Bellomo, F. Brezzi, Mathematics, complexity and multiscale features of large systems of self-propelled particles, *Math. Models Methods Appl. Sci.* **26** (2016) 207–214.
- [3] N. Bellomo, D. Knopoff, and J. Soler, On the difficult interplay between life, “complexity”, and mathematical sciences, *Math. Models Methods Appl. Sci.* **23** (2013) 1861–1913.
- [4] B. M. Bolker, S. W. Pacala, Using moment equations to understand stochastically driven spatial pattern formation in ecological systems, *Theoret. Population Biol.* **52** (1997) 179–197.
- [5] B. M. Bolker, S. W. Pacala, C. Neuhauser, Spatial dynamics in model plant communities: What do we really know? *The American Naturalist* **162** (2003) 135–148.
- [6] J. T. Cox, Coalescing random walks and voter model consensus times on the torus in  $\mathbb{Z}^d$ . *Ann. Probab.* **17** (1989) 1333–1366.
- [7] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. I (3rd Ed.). John Wiley & Sons, New York Chichester Brisbane Toronto, 1970.
- [8] D. Finkelshtein, M. Friesen, H. Hatzikirou, Yu. Kondratiev, T. Krüger, O. Kutoviy, Stochastic models of tumour development and related mesoscopic equations, *Interdisciplinary Studies of Complex Systems* **7** (2015) 5–85.
- [9] D. L. Finkelshtein, Yu. G. Kondratiev, O. Kutoviy, Individual based model with competition in spatial ecology, *SIAM J. Math. Anal.* **41** (2009) 297–317.
- [10] D. L. Finkelshtein, Yu. G. Kondratiev, O. Kutoviy, Semigroup approach to birth-and-death stochastic dynamics in continuum, *J. Funct. Anal.* **262** (2012) 1274–1308.
- [11] D. L. Finkelshtein, Yu. G. Kondratiev, Yu. Kozitsky, O. Kutoviy, The statistical dynamics of a spatial logistic model and the related kinetic equation, *Math. Models Methods Appl. Sci.* **25** (2015) 343–370.
- [12] H. Groemer, Some basic properties of packing and covering constants, *Discrete Comput. Geom.* **1** (1986) 183–193.
- [13] J. F. C. Kingman, *Poisson processes*. Oxford Studies in Probability, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.
- [14] Yu. G. Kondratiev, Yu. Kozitsky, The evolution of states in a spatial population model, *J. Dyn. Diff. Equat.* (2016)

- [15] Yu. Kondratiev, T. Kuna, Harmonic analysis on configuration space. I. General theory, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **5** (2002) 201–233.
- [16] Yu. Kondratiev, T. Kuna, M. J. Oliveira, Holomorphic Bogoliubov functionals for interacting particle systems in continuum, *J. Funct. Anal.* **238** (2006) 375–404.
- [17] Yu. Kozitsky, Dynamics of spatial logistic model: finite systems, in: J. Banasiak, A. Bobrowski, M. Lachowicz (Eds.), *Semigroups of Operators – Theory and Applications*: Będlewo, Poland, October 2013. Springer Proceedings in Mathematics & Statistics 113, Springer 2015, pp. 197–211.
- [18] D. J. Murrell, U. Dieckmann, R. Law, On moment closures for population dynamics in continuous space, *J. Theoret. Biol.* **229** (2004) 421–432.
- [19] C. Neuhauser, Mathematical challenges in spatial ecology, *Notices of AMS* **48** (11) (2001) 1304–1314.
- [20] A. North and O. Ovaskainen, Interactions between dispersal, competition, and landscape heterogeneity, *Oikos* **116** (2007) 1106–1119.
- [21] O. Ovaskainen, D. Finkelshtein, O. Kutovyi, S. Cornet, B. Bolker, Yu. Kondratiev, A general mathematical framework for the analysis of spatio-temporal point processes, *Theor. Ecol.* **7** (2014) 101–113.
- [22] L. M. Ricciardi, Stochastic population theory: birth and death processes, in: *Mathematical Ecology*, T. G. Hallam and S. A. Levin (Eds.) Springer-Verlag, Berlin, 1986, pp. 155–190.

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